DOCUMENT RESUME

ED 175 683 SE 028 671

AUTHOR Bridgess, M. Philbrick, Ed.
TITLE Supplementary and Enrichment Series: Absolute Value. Teachers' Commentary. SP-25.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 66
NOTE 48p.; For related documents, see SE 028 648-675; Contains occasional light and broken type

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Algebra; Curriculum; *Curriculum Guides; Enrichment; Graphs; *Instruction; Mathematics Education; *Number Concepts; Secondary Education; *Secondary School Mathematics
IDENTIFIERS *Complex Numbers; *School Mathematics Study Group

ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include addition and multiplication in terms of absolute value, graphs of absolute value in the Cartesian plane, absolute value and quadratic expressions, complex numbers, and vectors. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-25

SUPPLEMENTARY and ENRICHMENT SERIES

ABSOLUTE VALUE

Teachers' Commentary

Edited by M. Philbrick Bridgess

H56

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

CONTENTS

Chapter 1

INTRODUCTION

The usual definition of the absolute value of the real number n is:

$$|n| = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0. \end{cases}$$

This is also the form in which the absolute value is most commonly used. On the other hand, since students seem to have difficulty with definitions of this kind, we prefer to define the absolute value of a number in such a way that it can be clearly pictured on the number line. You must avoid at all costs allowing the student to think of absolute value as the number obtained by "dropping the sign". This way of thinking about absolute value, although it appears to give the correct "answer" when applied to specific numbers such as -3 or 3, leads to no end of trouble when variables are involved. Other less common names for absolute value are numerical value, magnitude, and modulus.

By observing that this "greater" of a number and its opposite is just the distance between the number and 0 on the Real Number line, we are able to interpret the absolute value "geometrically". In the Calculus, frequent use is made of the notion "the maximum of".

The symbolism $\sqrt{2}$ always denotes the positive number whose square is 2. The negative number whose square is 2 is written $-\sqrt{2}$.

Answers to Problem Set 1-1a

1. (a) 7; the greater of -7 and 7 is 7.
   (b) 3; the greater of -(-3) and its opposite -3 is -(-3) or 3.
   (c) 7; 6 - 4 + 5 is another name for 7 and 7 is greater than its opposite, -7.
   (d) 0; by the multiplication property of 0, (14 × 0) is 0, and the absolute value of 0 is 0.
   (e) 14; by the addition property of 0, (14 + 0) is 14, and 14 is greater than its opposite, -14.
   (f) 3; the opposite of the opposite of the opposite of 3 is simply -3 and the opposite of -3, 3, is greater than -3.

2. (a) non-negative
   (b) non-negative
   (c) non-negative
   (d) negative
   (e) non-negative
   (f) non-negative
   (g) non-negative

3. For the negative number x, $|x|$ is greater than x since, for x negative, $|x|$ is positive by problem 2(b). Since any negative number is less than any positive number, $x < |x|$ for all negative x.

4. The set $\{-1, -2, 1, 2\}$ is closed under the operation of taking the absolute value of its elements. Taking the absolute value of each element of the set,

$$|-1| = 1 \quad |-2| = 2 \quad |1| = 1 \quad |2| = 2,$$

we find that the set of absolute values of the numbers of the original set to be $\{1, 2\}$. Since $\{1, 2\}$ is a subset of $\{-2, -1, 1, 2\}$, this latter set is closed under the operation of taking absolute values of its elements.

5. Yes. It is a subset consisting of the non-negative reals.

Answers to Problem Set 1-1b

1. (a) $|-7| < 3$ or $7 < 3$. False
   (b) $|-2| \leq |-3|$ or $2 \leq 3$. True
   (c) $|4| < |1|$ or $4 < 1$. False
   (d) $2 \nless |-3|$ or $2 \nless 3$. False
   (e) $|-5| \nless |2|$ or $5 \nless 2$. True
   (f) $-3 < 17$. True
   (g) $-2 < |-3|$ or $-2 < 3$. True
   (h) $|\sqrt{16}| > |-4|$ or $4 > 4$. False
   (i) $|2|^2 = 4$ or $2^2 = 4$. True

   (b), (e), (f), (g) and (i) are true.

2. (a) $|2| + |3| = 2 + 3 = 5$.
   (b) $|-2| + |3| = 2 + 3 = 5$.
   (c) $-(|2| + |3|) = -(2 + 3) = -5$.
   (d) $-(|-2| + |3|) = -(2 + 3) = -5$.
   (e) $|-7| - (7 - 5) = 7 - 2 = 5$.
   (f) $7 - |-3| = 7 - 3 = 4$.
   (g) $|-5| \times 2 = 5 \times 2 = 10$.
   (h) $-(|-5| - 2) = -(5 - 2) = -3$.

(i) $|-3| - |2| = 3 - 2 = 1.$

(j) $|-2| + |-3| = 2 + 3 = 5.$

(k) $-(|-3| - 2) = -(3 - 2) = -1.$

(l) $-(|-2| + |-3|) = -(2 + 3) = -5.$

(m) $3 - |3 - 2| = 3 - 1 = 2.$

(n) $-(|-7| - 6) = -(7 - 6) = -1.$

(o) $|-5| \times |-2| = 5 \times 2 = 10.$

(p) $-(|-2| \times 5) = -(2 \times 5) = -10.$

(q) $-(|-5| \times |-2|) = -(5 \times 2) = -10.$

3. (a) $|x| = 1.$ The truth set is $\{1, -1\}$.

(b) $|x| = 3.$ The truth set is $\{3, -3\}$.

(c) $|x| + 1 = 4.$ The truth set is $\{3, -3\}$, the same as that of $|x| = 3$.

(d) $5 - |x| = 2.$ The truth set is $\{3, -3\}$.

(e) The truth set is ϕ.

(f) The truth set is $\{0\}$.

Some students may see these just by inspection. Others may think of the fact that $|x|$ is the distance on the number line from zero, so in (a), for instance, x must be 1 unit from zero. Therefore $x = 1$ or $x = -1$. Still others may reason as follows, for (d):

To find the truth set for $5 - |x| = 2$:

If $x \geq 0$, $|x| = x$, so $5 - x = 2$ or $x = 3$.

If $x < 0$, $|x| = -x$, so $5 - (-x) = 2$ or $x = -3$.

The truth set is $\{3, -3\}$.

Have several students explain their reasoning, as this is a splendid opportunity to check their understanding of absolute value.

4. Here students may have difficulty in finding a starting point. It may be helpful for them to refer back to Problem Set 1-1a, Problems 2 and 3.

(a) $|x| \geq 0$ is true for all real numbers x.

If $x \geq 0$, $|x| \geq 0$. See Problem Set 1-1a, Problem 2(b);

If $x < 0$, $|x| > 0$. See Problem Set 1-1a, Problem 3(b).

(b) $x \leq |x|$ is true for all real numbers x.

If $x \geq 0$, $x = |x|$.

If $x < 0$, $x < |x|$.

(c) $-x \leq |x|$ is true for all real numbers x.

If $x \geq 0$, $-x \leq |x|$.

If $x < 0$, $-x = |x|$.

(d) $-|x| \leq x$ is true for all real numbers x.

If $x \geq 0$, $-|x| \leq x$.

If $x < 0$, $-|x| = x$.

5. If x is negative, the absolute value of x, being the greater of the number and its opposite, is the opposite of x, that is

$$|x| = -x \quad \text{or} \quad -x = |x|.$$

Since $-x$ and $|x|$ are in this case names for the same number, their opposites also will be names for the same number, so that $-(-x) = -|x|$. As the opposite of the opposite of x is x, we may say $-(-x) = x$ and finally

$$x = -|x|.$$

If the teacher has not had much experience with graphs of inequalities, he would find the pamphlet on "Inequalities" very helpful.

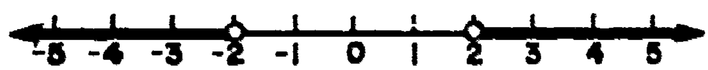
Answers to Problem Set 1-1c

1. Graph the truth sets of the following sentences

(a) $|x| < 4$

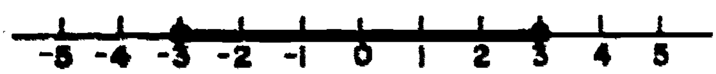
The student may arrive at the required graph by trial of different numbers for x, in the sentence. He may instead reason the exercise out somewhat as follows: The sentence states that "the absolute value of x is less than 4". On the number line, this statement becomes "x is less than 4 units from 0". Therefore, the graph of "$|x| < 4$" is the one given above.

(b) $|x| > 2$



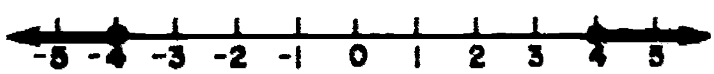
As in part (a) the student here may find the required set by trial and error, or by recalling the interpretation of absolute value as a distance on the number line.

(c) $|x| \leq 3$



Part (c) differs from part (a) in that the endpoints are included. Heavy black dots are used to indicate this.

(d) $|x| \geq 4$



(e) $|x| < -2$. ϕ. No graph. Absolute value of x cannot be negative.

(f) $|x| > 0$.

Truth set contains every real number except 0.

(g) $|x| > -1$.

Truth set is set of all real numbers.

2. (a) $|x| \leq 4$. Note that the endpoints are included.
   (b) $|x| \geq 1$. Note that the points whose coordinates are 1 and -1 are included. For the sake of brevity, we say "Points 1 and -1".
   (c) $|x| > 3$.

The distinction between "and" and "or" must be carefully observed. In a compound sentence, the use of "and" implies that both clauses are true, the use of "or" implies that one, or the other, or both clauses are true. This is the inclusive use of "or". If we are working with truth sets of open sentences, the use of "and" implies the intersection of the sets, the use of "or" implies the union of the sets. We can state this formally as follows:

<u>Intersection:</u> $P \cap Q = \{x : x \in P \wedge x \in Q\}$

(P <u>intersect</u> Q is the set of all numbers x such that x is an element of P <u>and</u> x is an element of Q)

<u>Union:</u> $P \cup Q = \{x : x \in P \vee x \in Q\}$

Answers to Problem Set 1-1d

1. (a) and
   (b) [illegible]
   (c) $|x| < \frac{5}{2}$. (Alternate answer: $-\frac{5}{2} < x < \frac{5}{2}$)

2. (C) and (E). (A) would be $|x| \geq 8$, (B) is the null set, (D) is the set of real numbers.

3. (B) and (D). (A) states that temperature was below 5°, (C) indicates all possible temperatures, (E) states that temperature was above 5°, <u>or</u> below -5°.

1-2. Subtraction, Distance and Absolute Value

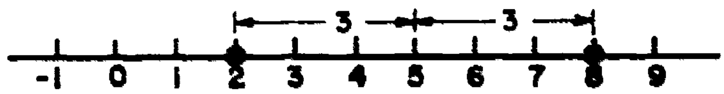
The relation between the difference of two numbers and the distance between these points on the number line is introduced here to make good use again of the number line to help illustrate our ideas.

You are no doubt aware, however, of the fact that $(a - b)$ as a directed distance and $|a - b|$ as a distance are very helpful concepts in dealing with slope and distance in analytic geometry.

Answers to Problem Set 1-2a

1. (a) $|3 - 7|$ (d) $|t - 6|$
   (b) $|6 - (-2)|$ (e) $|y - 0|$
   (c) $|(-8) - (-5)|$ (f) $|m - (-4)|$

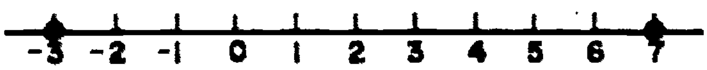
2. (a) The two numbers x such that the distance between x and 5 is 3 are the numbers corresponding to the two points 3 units away from 5



$\{2,8\}$

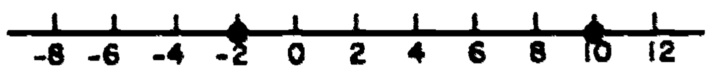
Though the above is the suggested approach to this problem, some students may do it by the definition of absolute value.
If $x - 5 \geq 0$, then $3 = |x - 5| = x - 5$ and $x = 8$;
if $x - 5 < 0$, then $3 = |x - 5| = -(x - 5) = -x + 5$, from which we get $x = 2$.

(b)



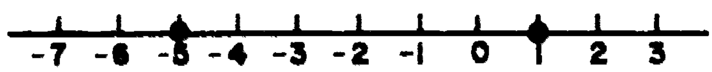
$\{-3,7\}$

(c)



$\{-2,10\}$

(d)



$\{-5,-1\}$

(e) ϕ. Absolute value is never negative.

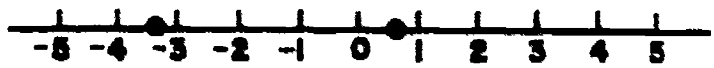
(f)

$\{-7,-3\}$

(g)

$\{1,2\}$

The sentence states that the distance between twice some number and 3 is 1. Twice the number would be 2 or 4, hence, the number is 1 or 2. By definition, also, if $2x - 3 \geq 0$, then $2x - 3 = 1$ and $x = 2$, if $2x - 3 < 0$, then $-(2x - 3) = 1$, and $x = 1$.

(h) 

$\{-\frac{10}{3}, \frac{2}{3}\}$

Answers to Problem Set 1-2b

1. (a) The truth set of the sentence $|x - 2| < 2$ is the set of numbers satisfying the inequality $0 < x < 4$. We can state this:
$\{x : |x - 2| < 2\} = \{x : 0 < x < 4\}$.

Rather than use formal methods for the solution of the inequality, the student will be guided by the question: What is the set of numbers x such that the distance between x and 2 is less then 2? As in the case of the preceding problem set, however, the student may work directly from the definition of absolute value instead of the suggested approach.

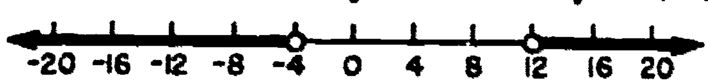
For example, if $x - 2 \geq 0$, then $|x - 2| = x - 2$ and

$$x - 2 < 2$$
$$x < 4 .$$

If $x - 2 < 0$, then $|x - 2| = -(x - 2) = -x + 2$ and

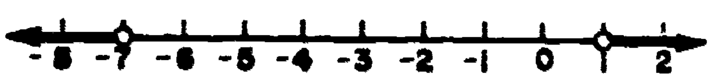
$$-x + 2 < 2$$
$$-x < 0$$
$$x > 0$$

(b) The set of all numbers y such that $y < -4$ or $y > 12$

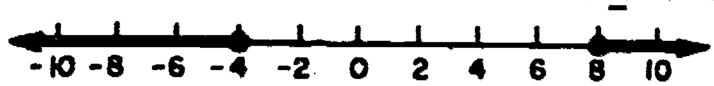


(c) The set of all numbers r such that $-12 < r < -2$

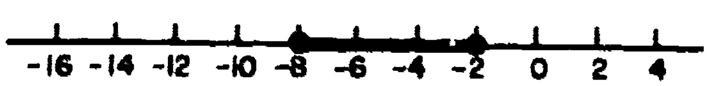
(d) The set of all numbers b such that $b < -7$ or $b > 1$



(e) The set of all numbers c such that $c \leq -4$ or $c \geq 8$



(f) The set of all numbers d such that $-8 \leq d \leq -2$



(g) Since $|x| + 3 > 0$ is equivalent to $|x| > -3$, the truth set is the set of real numbers and the graph is the number line.

(h) Since $|x - 4| + 5 < 0$ is equivalent to $|x - 4| < -5$, the truth set is ϕ. The graph can be shown by an unmarked line with a "ϕ" at the end.

2. (A), (B), and (D) are equivalent. (C) is <u>not</u> equivalent, since it is the set of all real numbers.

3. The graphs are all the same.

4. (a) If $|x| = x$, then $x \geq 0$.

   (b) Since $-|m| < 2$ is equivalent to $|m| > -2$, then m is the set of real numbers.

   (c) ϕ

   (d) If $|u| = -u$, then $u \leq 0$.

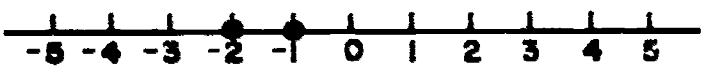
   (e) If $|v| \geq 0$, then v is the set of real numbers.

5. (a)

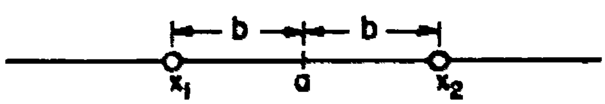
(b)

(c)

(d)



6.



Since the distance between x_1 and a and also between x_2 and a is b, it is apparent that $x_1 = a - b$ and $x_2 = a + b$. Therefore, $a - b < x < a + b$. This assumes that b is non-negative.

7. If $|x - 2| < \frac{1}{4}$, then $\frac{7}{4} < x < \frac{9}{4}$.

   If $x > \frac{7}{4}$, then $5x + 3 > \frac{47}{4}$. $\therefore$ $|(5x + 3) - 13| < \frac{5}{4}$.

   If $x < \frac{9}{4}$, then $5x + 3 < \frac{57}{4}$. $\therefore$ $|(5x + 3) - 13| < \frac{5}{4}$.

8. If $|x + 2| < 5$, then $-7 < x < 3$. Thus, $0 \leq x^2 < 49$.

Chapter 2

ADDITION AND MULTIPLICATI ›N IN TERMS OF ABSOLUTE VALUE

In Section 2-1, a precise definition of addition is formulated, first in English and then in the language of algebra. Section 2-2 is concerned with multiplication. It is difficult to find "real life" situations which will suggest what multiplication involving negative numbers ought to be. After we have addition, however, insistence on the distributive property suggests how multiplication must be defined. Section 2-3 on the us of absolute value symbol when simplifying radicals follows appropriately after multiplication.

2-1. Definition of Addition

Our immediate objective is more ambitious than just to teach the arithmetic of negative numbers. We want to bring out the important fact that what is really involved here is an extension of the operation of ·ddition from the numbers of arithmetic (where the operation is familiar) to all real numbers in such a way that the basic properties of addition are preserved. This means that we must give a definition of addition in terms of only non-negative numbers and familiar operations on them. The result in the language of algebra is a formula for $a + b$ involving the familiar operations of addition, subtraction and opposite applied to the non-nega:ive $|a|$ and $|b|$. The complete formula appears formidable because of the variety of cases. The idea is simple, however; and is nothing more than a general statement of exactly what we always do in obtaining the sum of negative numbers.

The main problem is to lead up to the general definition of $a + b$ in a plausible way. The case of the sum of two negative numbers is considered in some detail first. Then the other cases are considered more briefly leading up to the general definition first in English and then in the language of algebra.

Answers to Problem Set 2-1

1. (a) difference
   (b) greater
   (c) absolute value
2. (a) Yes
   (b) Yes

3. This exercise brings out an important relationship which we shall see again in Section 5-1 on Absolute Value of Complex Numbers. If we had a section on the addition of vectors, it would also appear there.

(a) Yes

(b) Yes

(c) No, $5 \neq 1$

(d) No, $5 \neq 1$

(e) The left member is greater than the right member.

(f) $|a| + |b| \geq |a + b|$

4. Here are some possibilities

(a) $|9 - 2| = |9| - |2|$

(b) $|2 - 9| > |2| - |9|$

(c) $|9 - (-2)| > |9| - |-2|$

(d) $|(-2) - 9| > |-2| - |9|$

(e) $|(-9) - 2| > |-9| - |2|$

(f) $|2 - (-9)| > |2| - |-9|$

(g) $|(-9) - (-2)| = |-9| - |-2|$

(h) $|(-2) - (-9)| > |-2| - |-9|$

From similar examples the student will, we hope, infer that for any real numbers a and b,

$$|a - b| \geq |a| - |b|,$$
$$|a - b| \geq |b| - |a|.$$

5. (a) The student will probably use numerical examples. By definition, however, we can prove it.

If $a = b$, then $|a - b| = |b - a|$, since $|0| = |0|$.

If $a \neq b$, then $a - b > 0$ or $a - b < 0$.

If $a - b > 0$,

Then $|a - b| = a - b$ but $b - a < 0$, so $|b - a| = -(b - a) = a - b$.

Hence, $|a - b| = |b - a|$.

The proof is similar for $a - b < 0$.

(b) The numerical examples from Exercise 4 could be used. We could prove this as follows:

$$|b| - |a| = -(|a| - |b|),$$

so that we now have

$$|a - b| \geq |a| - |b|,$$
$$|a - b| \geq -(|a| - |b|).$$

In other words, if $|a| \neq |b|$, then $|a - b|$ is greater than or equal to the larger of $|a| - |b|$ and its opposite $-(|a| - |b|)$, and this means that

$$|a - b| \geq \left| |a| - |b| \right|;$$

if $|a| = |b|$, then $\left| |a| - |b| \right| = |0| = 0$, so that the latter inequality is also true in this case.

The distance between a and b is found to be at least as great as the distance between $|a|$ and $|b|$, because a and b can be on opposite sides of 0 while $|a|$ and $|b|$ must be on the same side.

6. $|a - b| + |b| \geq |a|$ Addition Property of Opposites
   $|a - b| + |b| + (-|b|) \geq |a| + (-|b|)$ Addition Property of Order

---

2-2. Definition of Multiplication

There are several ways of making multiplication of real numbers seem plausible. It seems best to let the choice of definition of multiplication be a necessary outgrowth of a desire to retain the distributive property for real numbers.

Make clear to the students that what we are doing here is not a proof. We couldn't prove anything about something which has not been defined. However, to guide us in choosing the definition we ask "If we had a definition of the product ab for negative numbers, how would the numbers behave under the distributive property?" We find that they would behave in such a way that

$$0 = 6 + (2)(-3)$$

would have to be true. But if the uniqueness of the additive inverse is to continue to hold, $(2)(-3)$ would then have to be the opposite of 6.

As in the case of addition, the point of view here is that we extend the operation of multiplication from the numbers of arithmetic to all real numbers so as to preserve the fundamental properties. This actually forces us to define multiplication in the way we have. In other words it could not be done in any other way without giving up some of the properties.

The general definition of multiplication for real numbers is stated in terms of absolute values because $|a|$ and $|b|$ are numbers of arithmetic, and we already know how to multiply numbers of arithmetic. The only problem for real numbers is to determine whether the product is positive or negative.

Answers to Problem Set 2-2

1. $|a||b| = |ab|$

The better students will be able to follow the proof of this:

Exactly one of the following is true:

$ab = |a| \cdot |b|$ or $ab = -(|a| \cdot |b|)$	Definition of the product of two real numbers.
(Note that the two possible values of ab are opposites.)	
Then $|ab|$ is either $|a| \cdot |b|$ or $-(|a| \cdot |b|)$, whichever is greater	Definition of absolute value.
But $|a| \cdot |b|$ is positive.	Definition of the product of two real numbers.
and $-(|a| \cdot |b|)$ is negative.	Definition of opposites.
Hence $|ab| = |a| \cdot |b|$.	A positive number is greater than a negative number.

2. If $n > 0$, then we know it is true from arithmetic.

If $n < 0$, then $\left|\frac{1}{n}\right| = -\frac{1}{n}$ and $\frac{1}{|n|} = \frac{1}{-n} = -\frac{1}{n}$. Hence, it is true, provided $n \neq 0$.

3. If $m = 0$, $n \neq 0$, then $\left|\frac{m}{n}\right| = 0$ and $\frac{|m|}{|n|} = 0$.

If $m > 0$, $n > 0$, then $\left|\frac{m}{n}\right| = \frac{m}{n}$ and $\frac{|m|}{|n|} = \frac{m}{n}$.

If $m > 0$, $n < 0$, then $\left|\frac{m}{n}\right| = -\frac{m}{n}$ and $\frac{|m|}{|n|} = \frac{m}{-n} = -\frac{m}{n}$.

If $m < 0$, $n > 0$, then $\left|\frac{m}{n}\right| = -\frac{m}{n}$ and $\frac{|m|}{|n|} = \frac{-m}{n} = -\frac{m}{n}$.

If $m < 0$, $n < 0$, then $\left|\frac{m}{n}\right| = \frac{m}{n}$ and $\frac{|m|}{|n|} = \frac{-m}{-n} = \frac{m}{n}$.

4.

$a \cdot 1 = -(|a||1|)$	Definition of Multiplication of real numbers.
$|a| = -a$	Definition of Absolute Value
$-|a| = -(-a) = a$	Opposite of the opposite of a number is the number itself.

5. If $a = 0$, $|a| = |-a|$, since $0 = -0$.

If $a > 0$, $|a| = a$, $-a < 0$. Thus, $|-a| = -(-a) = a$.

If $a < 0$, $|a| = -a$, $-a > 0$. Thus, $|-a| = -a$.

Which completes the proof.

6. If $a = 0$, $-|0| = 0 = |0|$. Thus, $-|a| = a = |a|$.
If $a > 0$, then $-|a| = -a$ and $-a < 0$. $|a| = a$. Thus,
$-|a| < a = |a|$.
If $a < 0$, then $-|a| = -(-a) = a$ and $-a > 0$. $|a| = -a$. Thus,
$-|a| = a < |a|$.
Hence, $-|a| \leq a \leq |a|$.

7. (a) $|x - 2| = x - 2$ if $x - 2 \geq 0$
(b) $|x - 2| = 2 - x$ if $x - 2 \leq 0$

8. In Section 4-1, we shall solve these by a different method. For the time being, the emphasis is on the definition.

(a) If $x + 3 \geq 0$, then $|x + 3| = x + 3$. Thus, $x + 3 = x$. Truth set is ϕ.
If $x + 3 < 0$, then $|x + 3| = -(x + 3)$. Thus, $-(x + 3) = x$. This gives $x = -\frac{3}{2}$; $-\frac{3}{2} + 3 \nless 0$. Truth set is ϕ.

(b) If $x - 2 \geq 0$, then $|x - 2| = x - 2$. Hence, $x - 2 = 2x + 5$. $x = -7$.
But $-7 - 2 \ngeq 0$. Thus, truth set is ϕ.
If $x - 2 < 0$, then $|x - 2| = -(x - 2)$. Hence, $-(x - 2) = 2x + 5$.
$x = -1$.
Since, $-1 - 2 < 0$, truth set is $\{-1\}$.

(c) $\{\frac{1}{5}, -9\}$

(d) $\{-1, 1, 3, 5\}$

(e) $(|x - 3| - 1)((|x - 3|)^2 + |x - 3| + 1) = 0$ $\{2,4\}$

(f) Truth set of $|x| = 2$ is $\{2,-2\}$.
Truth set of $|x - 5| = 3$ is $\{2,8\}$.
Truth set of compound sentence with "and" is $\{2\}$.

(g) Truth set of compound sentence with "or" is $\{-5, -1, 3\}$.

9. $ab = |a||b|$ — Definition of multiplication.
$ab = -(|a||b|)$
$= -(|b||a|)$ — Commutative property of multiplication for non-negative real numbers.
$= ba$ — Definition of multiplication.

18

10. (a) Use Exercise 1 above. Since $|a||b| = |ab|$,
$|mx + ma| = |m(x + a)| = |m||x + a|$.

(b) $|3x - 12| < d$ implies $|3||x - 4| < d$ or $|x - 4| < \frac{d}{3}$.

Therefore, x is the set of all real numbers such that
$4 - \frac{d}{3} < x < 4 + \frac{d}{3}$.

2-3. Absolute Value and Simplification of Radicals

There is little confusion over the symbols $\sqrt{9}$, $\sqrt{-9}$, and $-\sqrt{9}$; but as soon as a variable appears under a square root sign we must be careful. The difficulties come from the fact that the square root symbol always indicates the positive square root and that the radicand must be non-negative. Consider $\sqrt{a^2}$. If a is positive or negative, a^2, the radicand, is positive. Thus, $\sqrt{a^2} = a$ is true if a is positive but false if a is negative. The true equation is

$$\sqrt{a^2} = |a|.$$

Examples are:

$$\sqrt{9x^2} = 3|x|$$
$$\sqrt{(-3)^2} = |-3| = 3$$
$$\sqrt{(a + b)^2} = |a + b|.$$

Suppose we have a radical such as $\sqrt{16a}$. In this expression a must be non-negative; the definition of a root requires it. If a were negative we would have what is defined as an imaginary number, which we do not consider here. Some examples are

$$\sqrt{9x^3} = \sqrt{9x^2 \cdot x} = 3|x|\sqrt{x},$$

but since x is understood to be positive,

$$3|x|\sqrt{x} = 3x\sqrt{x}.$$
$$\sqrt{(x - 1)^3} = |x - 1|\sqrt{x - 1} = (x - 1)\sqrt{x - 1},\ x \geq 1.$$

Here is another interesting misuse of absolute value. Consider the following "proof" that all numbers are equal. If a and b are any real numbers, then

$$|a - b| = |b - a|,$$
$$(a - b)^2 = (b - a)^2,$$
$$a - b = b - a,$$

$$2a = 2b,$$
$$a = b.$$

Which step of this "proof" is faulty, and why?

The trouble with the "proof" can be best explained by inserting another step.

$$(a - b)^2 = (b - a)^2$$
$$\sqrt{(a - b)^2} = \sqrt{(b - a)^2}$$
$$|a - b| = |b - a|,$$

because $\sqrt{x^2} = |x|$. $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

If $a - b > 0$, then

$$b - a < 0.$$
$$a - b = -(b - a)$$
$$a - b = -b + a$$
$$a - b = a - b$$

If $a - b < 0$, then

$$b - a > 0.$$
$$-(a - b) = b - a$$
$$-a + b = b - a$$
$$b - a = b - a$$

Answers to Problem Set 2-3

1. (a) $2\sqrt{6}|x|$

   (b) $2x\sqrt{6x}$, and $x \geq 0$

   (c) $2x^2\sqrt{6x}$, and $x \geq 0$

2. (a) $4a^2\sqrt{2}$

   (b) $2a\ \sqrt[3]{4a}$

   (c) $2|a|\ \sqrt[4]{2}$

3. (a) $|x|\sqrt{x^2 + 1}$

   (b) $|x^3|$

   (c) $x^2 + |x|$

4. (a) $30x\sqrt{2}$, and $x \geq 0$

   (b) $3|x|y\sqrt{ay}$, and $a \geq 0$, and $y \geq 0$

5. (a) $\frac{|x|}{3}$

   (b) $\frac{2}{3|y|}$, and $y \neq 0$

   (c) $\frac{\sqrt{y}}{x}$ and $x > 0$ and $y \geq 0$

6. (a) $\frac{|a|\sqrt{3x}}{5x}$ and $x > 0$

   (b) $\frac{a^3}{15}\sqrt{5}$ and $a \geq 0$

7. (a) $2|a + b|$

   (b) $\frac{|x - 2|}{3}$

   (c) $|c - 3|$

8. $2|a|\sqrt{3}$

9. $\frac{2|m|\sqrt{q}}{q} + 7|m|q\sqrt{2q}$ and $q > 0$

10. $\sqrt{(2x - 1)^2} = |2x - 1|$

   (a) $x < \frac{1}{2}$, $2x < 1$, $2x - 1 < 0$, $|2x - 1| = -(2x - 1) = 1 - 2x$

   (b) $x > \frac{1}{2}$, $2x > 1$, $2x - 1 > 0$, $|2x - 1| = 2x - 1$

   (c) $x = \frac{1}{2}$, $2x - 1 = 0$, $|2x - 1| = 0$

Chapter 3

GRAPHS OF ABSOLUTE VALUE IN THE CARTESIAN PLANE

This chapter dealing with absolute value is valuable not only for the opportunity it provides for recall of work done with absolute value earlier in the pamphlet, but also for the opportunity it provides for examining what happens to a graph when certain changes are made in its equation.

Answers to Problem Set 3-1a

1. (a) $k > 0$ (b) $k = 0$

2. (a) If $k > 0$, then there are two horizontal lines, one, k units above the x-axis, the other, k units below the x-axis.
   (b) If $k = 0$, the graph is the x-axis.
   (c) If $k < 0$, there is no graph.

3. (a)

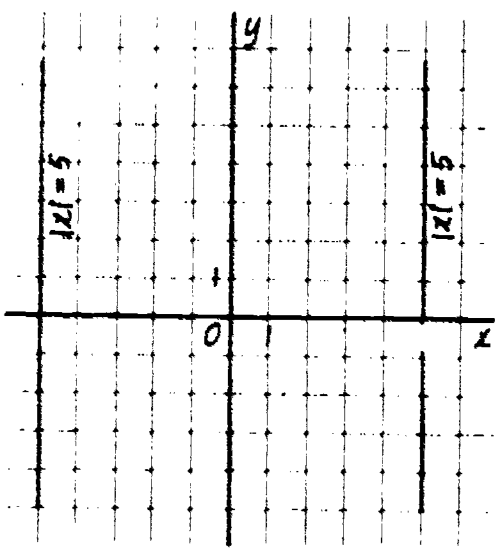


Figure for Problem 3(a)

(b)

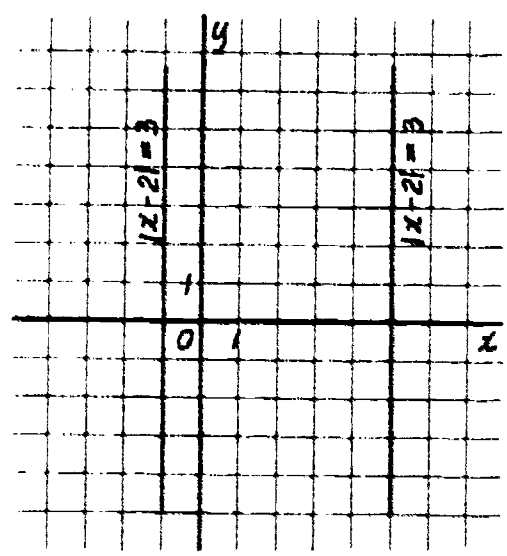


Figure for Problem 3(b)

(c)

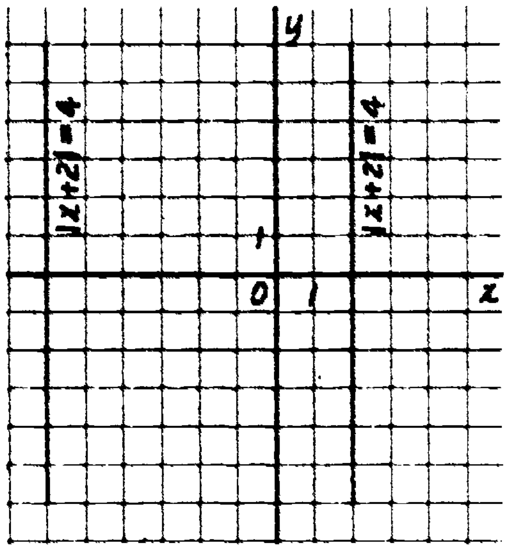


Figure for Problem 3(c)

(d)

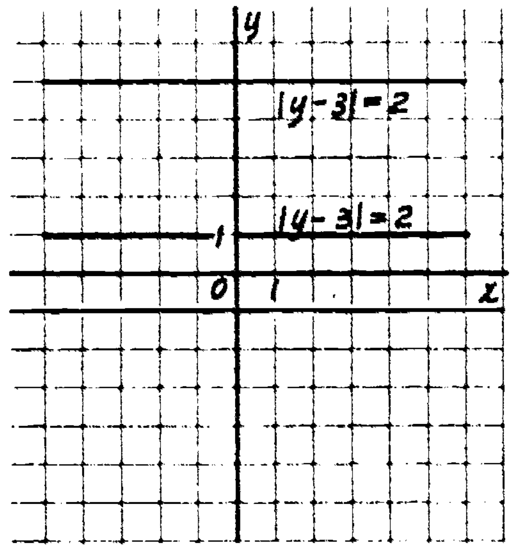


Figure for Problem 3(d)

(e)

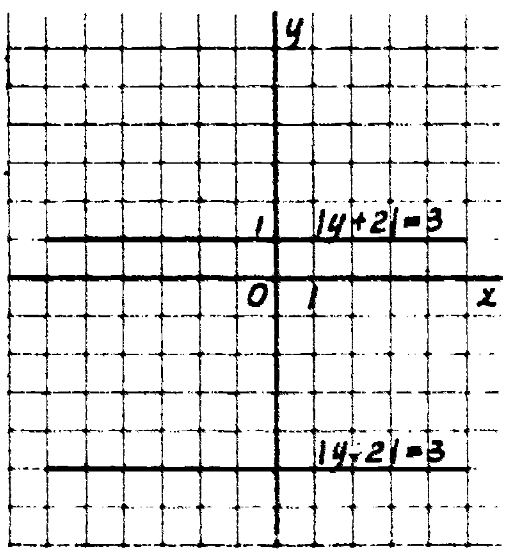


Figure for Problem 3(e)

(f)

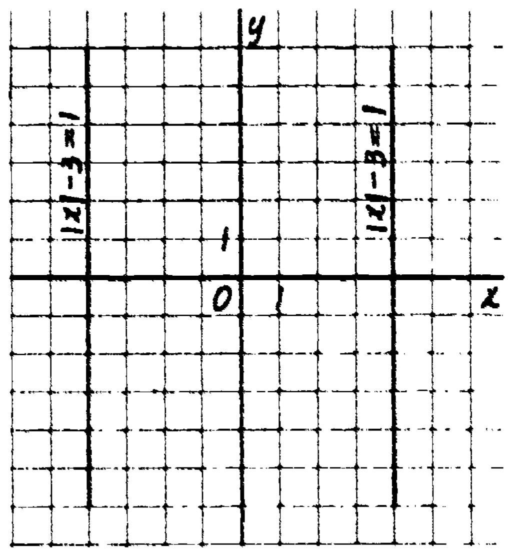


Figure for Problem 3(f)

(g) No graph. ϕ

(h) The entire quadrants I and IV, y-axis and positive portion of x-axis.

2

Answers to Problem Set 3-1b

1. (a)

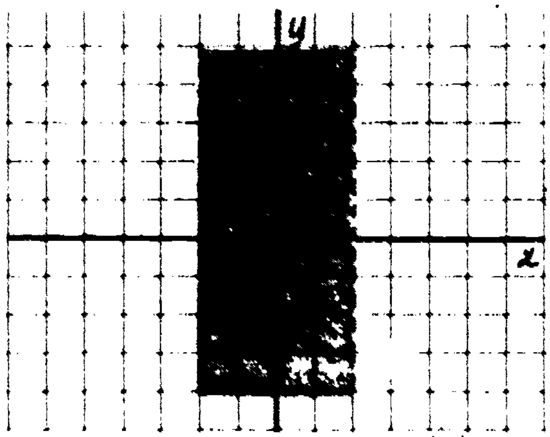


Figure for Problem 1(a)

(d)

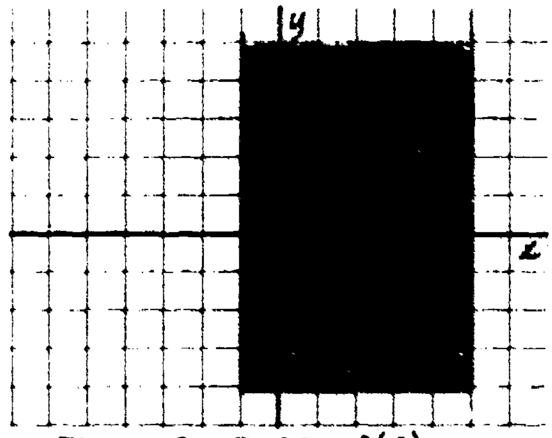


Figure for Problem 1(d)

(b)

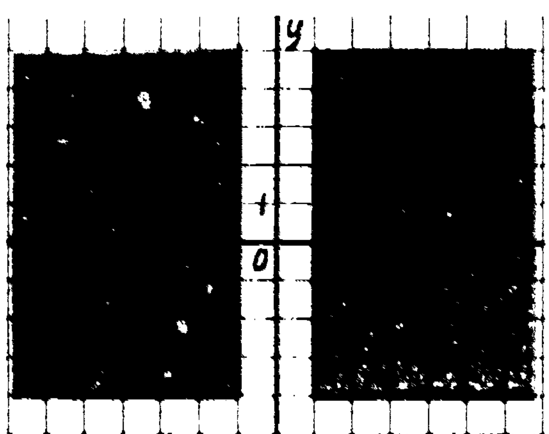


Figure for Problem 1(b)

(e)

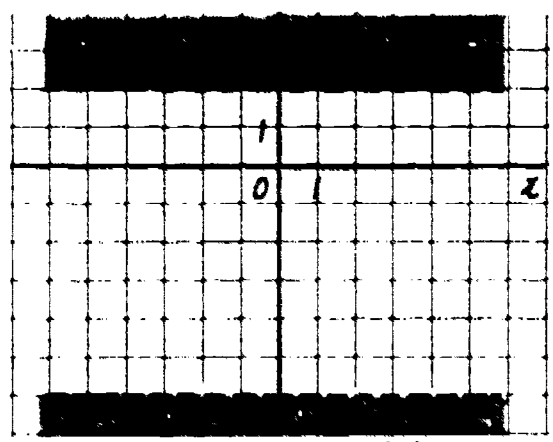


Figure for Problem 1(e)

(c)

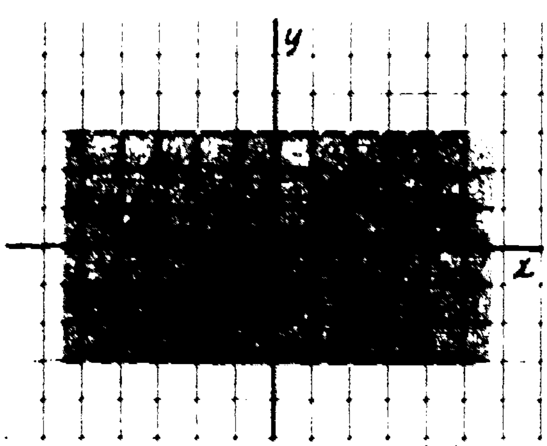


Figure for Problem 1(c)

(f) The graph would consist of the entire number plane except for points on the line, $x = -1$.

2. (a) The graph would consists of all points in Quadrants II and III, negative portions of x-axis but not the y-axis.

   (b) Since the truth set of $|x| < x$ is ϕ, there would be no points in its graph.

3-2. Graphs of Open Sentences with Two Variables

Whether x is positive or negative, the absolute value of x must be positive. So every value of y is positive for every value of x except 0. For $x = 0$, $y = 0$.

A simple equation whose graph would be two lines which do not form a right angle would be "$y = 2|x|$" or "$y = -2|x|$", or any equation of the form "$y = k|x|$" where k is not 1.

Answers to Problem Set 3-2a

1. (a)

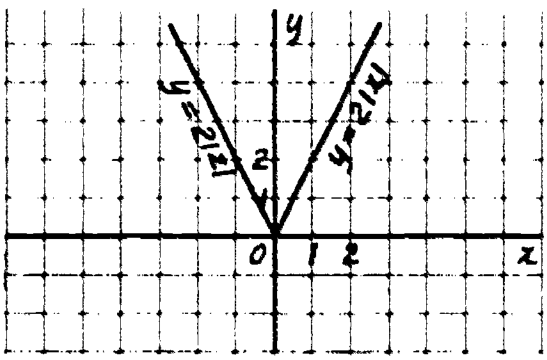


Figure for Problem 1(a)

(c)

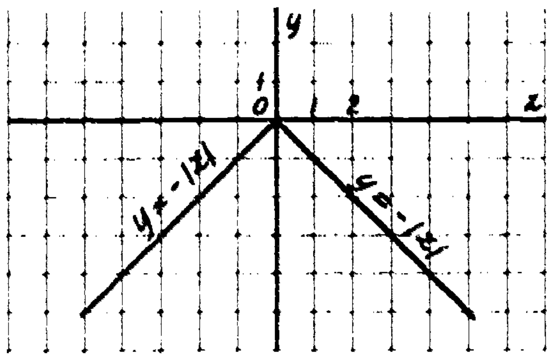


Figure for Problem 1(b)

(b)

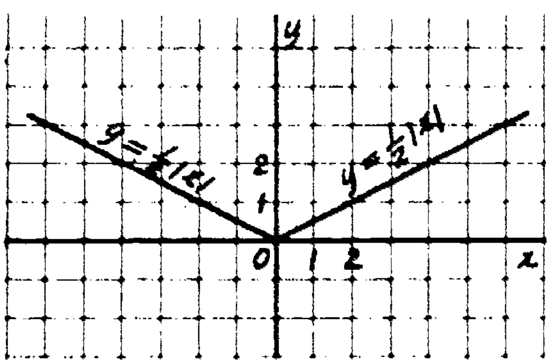


Figure for Problem 1(b)

(d)

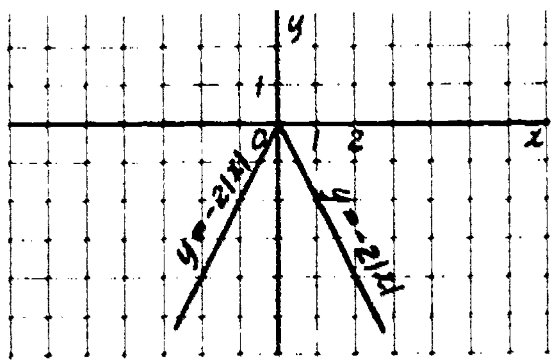


Figure for Problem 1(d)

(e)

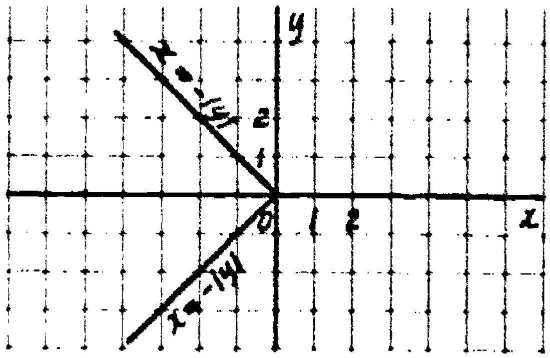


Figure for Problem 1(e)

(f)

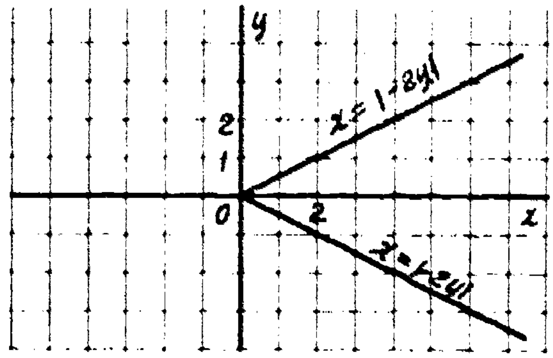


Figure for Problem 1(f)

2. (a)

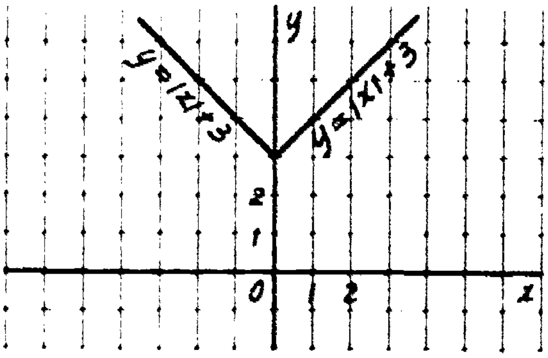


Figure for Problem 2(a)

(d)

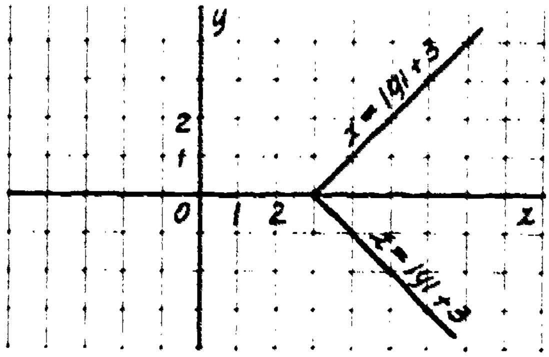


Figure for Problem 2(d)

(b)

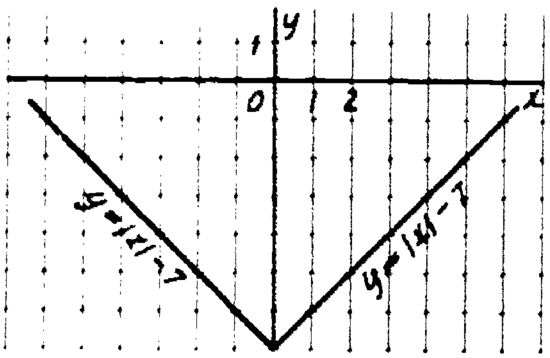


Figure for Problem 2(b)

(e)

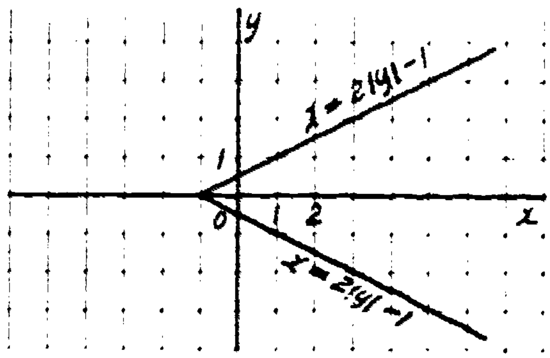


Figure for Problem 2(e)

(c)

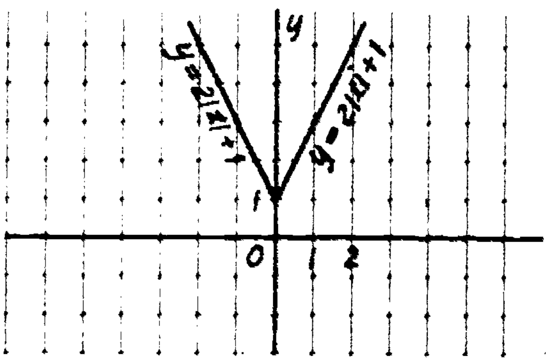


Figure for Problem 2(c)

(f)

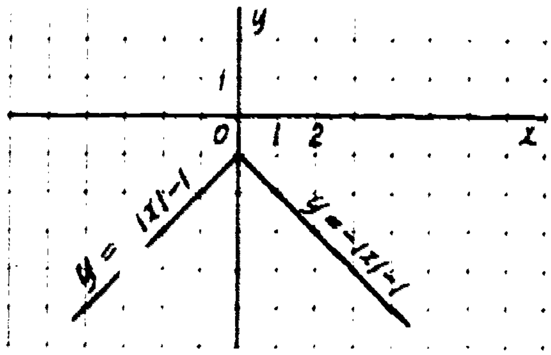


Figure for Problem 2(f)

3. (a)

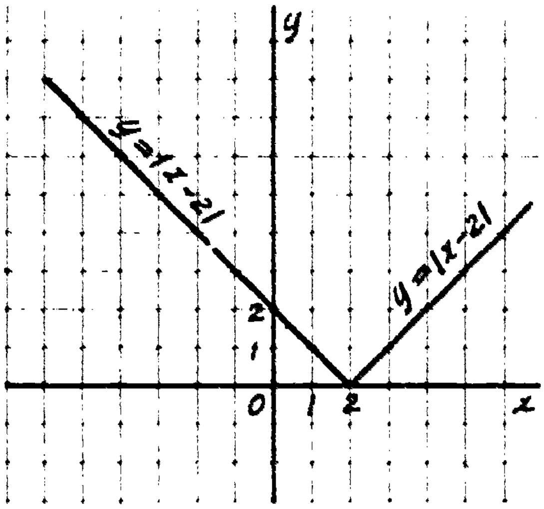


Figure for Problem 3(a)

(d)

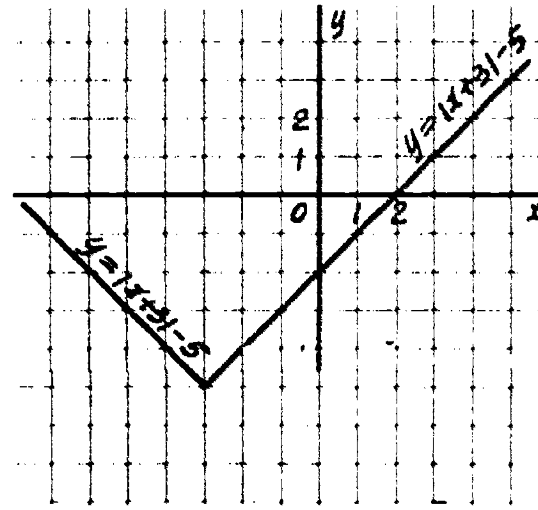


Figure for Problem 3(d)

(b)

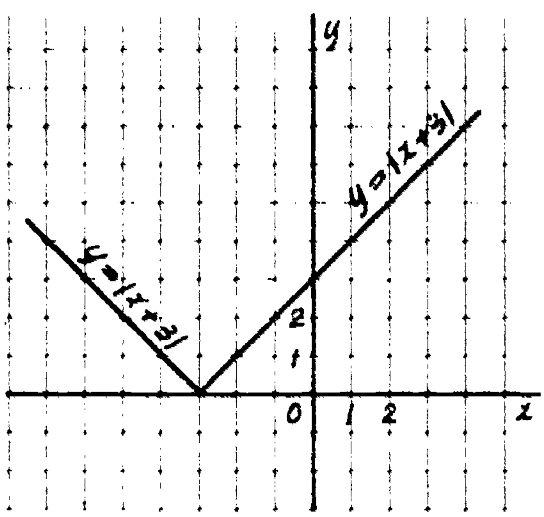


Figure for Problem 3(b)

(e)

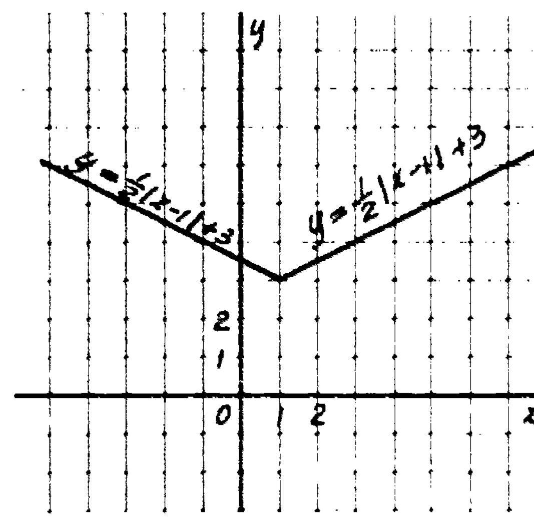


Figure for Problem 3(e)

(c)

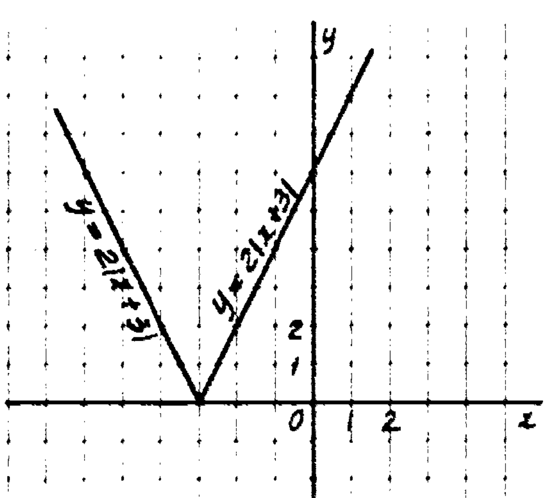


Figure for Problem 3(c)

4. Problem

1(c) The graph of "$y = -|x|$" can be obtained by revolving the graph of "$y = |x|$" $\frac{1}{2}$ revolution about the x-axis.

1(e) The graph of "$x = -|y|$" can be obtained by revolving the graph of "$x = |y|$" $\frac{1}{2}$ revolution about the y-axis.

2(a) The graph of "$y = |x| + 3$" can be obtained by sliding the graph of "$y = |x|$" up three units.

2(b) The graph of "$y = |x| - 7$" can be obtained by sliding the graph of "$y = |x|$" down seven units.

2(d) The graph of "$x = |y| + 3$" can be obtained by sliding the graph of "$x = |y|$" to the right three units.

2(f) The graph of "$y = -|x| - 1$" can be obtained by revolving the graph of "$y = |x|$" about the x-axis and then sliding it down one unit.

3(a) The graph of "$y = |x - 2|$" can be obtained by sliding the graph of "$y = |x|$" to the right two units.

3(b) The graph of "$y = |x + 3|$" can be obtained by sliding the graph of "$y = |x|$" to the left three units.

3(d) The graph of "$y = |x + 3| - 5$" can be obtained by sliding the graph of "$y = |x|$" to the left three units and down five units.

5. (a)

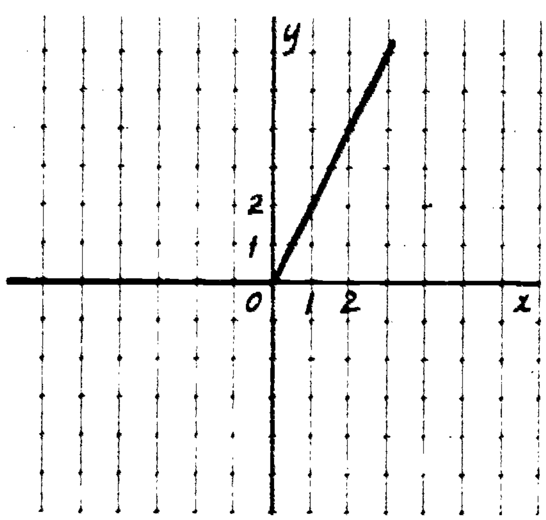


Figure for Problem 5(a)

(b)

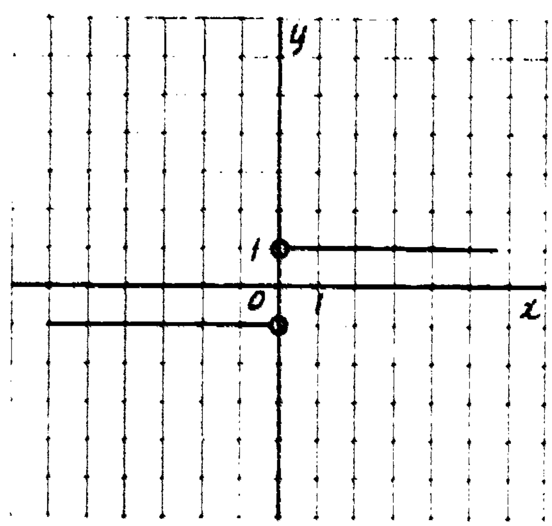


Figure for Problem 5(b)

6. (a)

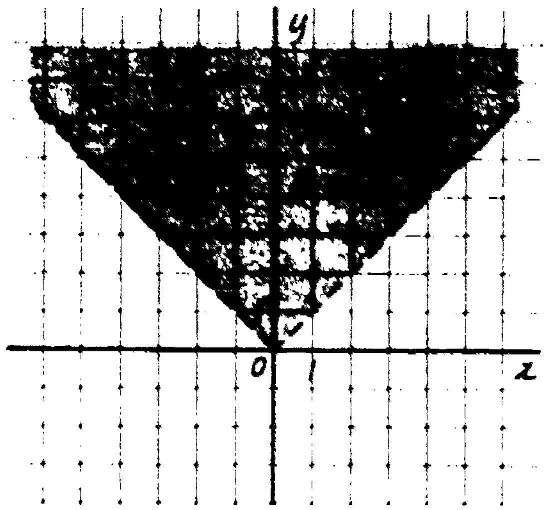


Figure for Problem 6(a)

(c)

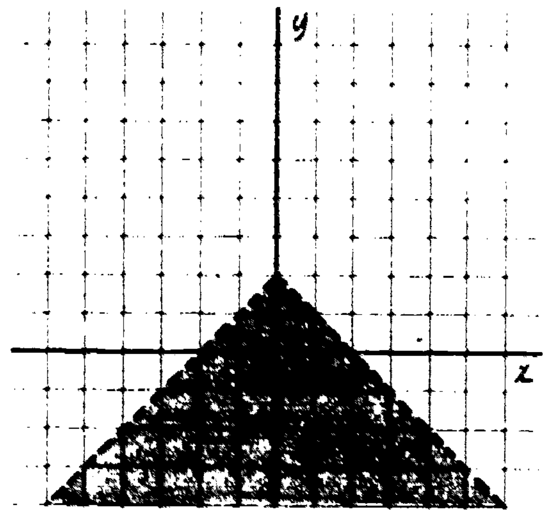


Figure for Problem 6(c)

(b)

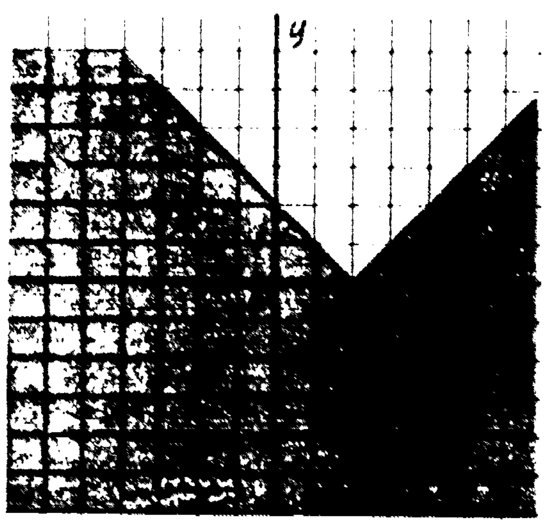


Figure for Problem 6(b)

(d)

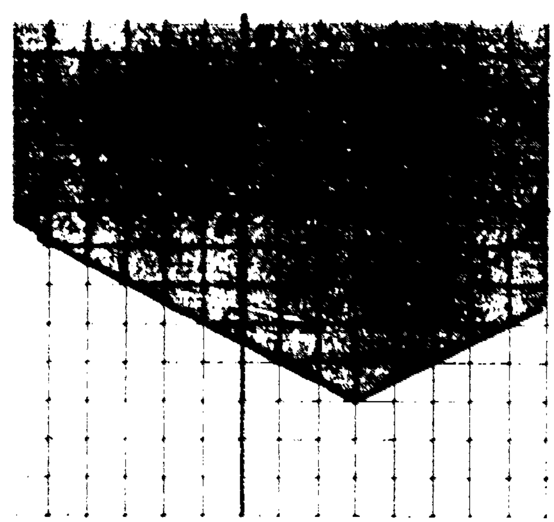


Figure for Problem 6(d)

If $x = 6$, there are no possible values of y, since $|y| = -1$ and this is impossible. If $y = |2|$, there are two possible values for y: $y = -2$ and $y = 2$.

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
|x|	5	3	3	1	1	0	0	1	1	3	3	5
|y|	0	2	2	4	4	5	5	4	4	2	2	0
y	0	2	-2	4	-4	5	-5	4	-4	2	-2	0

Answers to Problem Set 3-2b

1. The graph shown is the graph of $|x| + |y| = 5$, as well as the graphs of the four open sentences:

$x + y = 5$, and $0 \le x \le 5$
or $x - y = 5$, and $0 \le x \le 5$
or $-x + y = 5$, and $-5 \le x \le 0$
or $-x - y = 5$, and $-5 < x < 0$

It was necessary in these to limit the values of x so that only the indicated segments of the lines would be included.

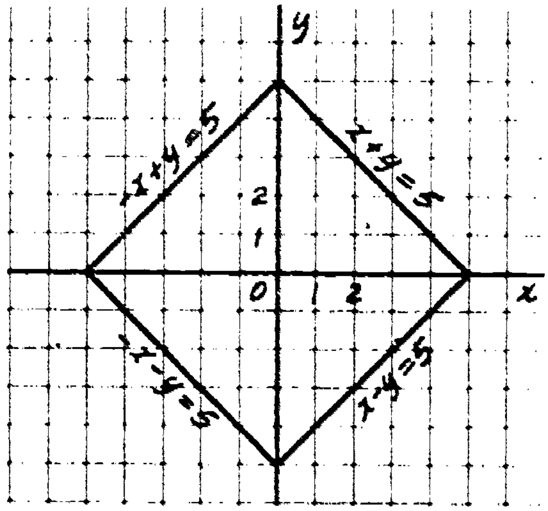


Figure for Problem 1

2. (a) Point out to the pupil the four open sentences implied here:

$x + y > 5$,
or $x - y > 5$,
or $-x + y > 5$,
or $-x - y > 5$.

The graphs of the corresponding equations: "$x + y = 5$ or $x - y = 5$ or etc." are then drawn with dotted lines. Now note that:

$x + y > 5$ becomes $y > -x + 5$
$-x + y > 5$ becomes $y > x + 5$

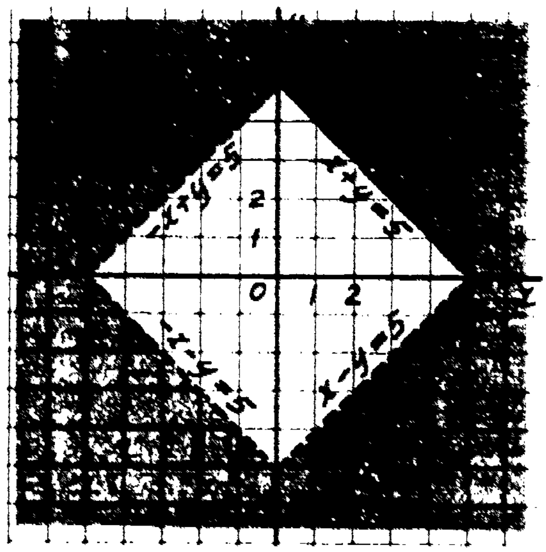


Figure for Problem 2(a)

So the area above each of the lines where "$x + y = 5$" and "$-x + y = 5$" is shaded. Also:

$x - y > 5$ becomes $y < x - 5$
$-x - y > 5$ becomes $y < -x - 5$

So the area below each of the lines where "$x - y = 5$" and "$-x - y = 5$" is shaded. Therefore the graph of $|x| + |y| > 5$ is all of the plane outside the graph of $|x| + |y| = 5$.

(b) In the same line of reasoning $|x| + |y| < 5$ implies:

$$x + y < 5,$$
and $x - y < 5,$
and $-x + y < 5,$
and $-x - y < 5.$

Hence, the graph is the area <u>inside</u> the graph of

$$|x| + |y| = 5.$$

Verify on the number line that "$|y| < k$" is equivalent to "$y < k$ <u>and</u> $-y < k$", whereas "$|y| > k$" is equivalent to "$y > k$ <u>or</u> $-y > k$".

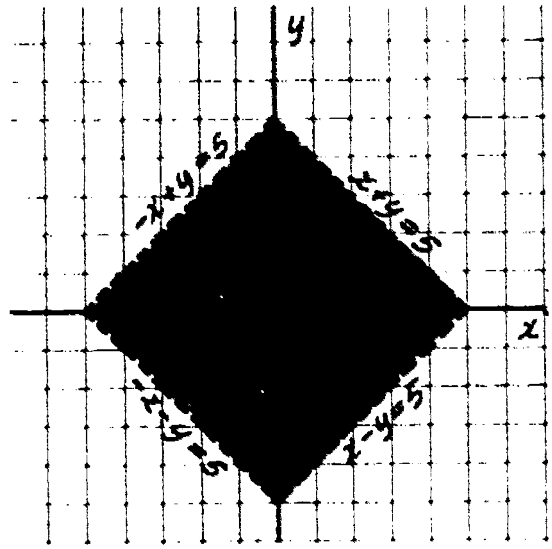


Figure for Problem 2(b)

(c) The graph is the same as that for (b), except that the lines are solid to indicate that the graph of

$$|x| + |y| = 5$$

is included, as well as the graph of $|x| + |y| < 5$.

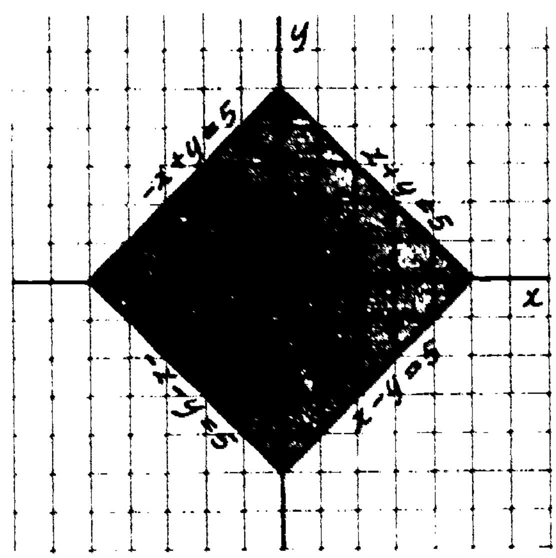


Figure for Problem 2(c)

(d) "$|x| + |y| \nless 5$" implies "$|x| + |y| = 5$ or $|x| + |y| > 5$", so the graph is the same as that for (a), except that the lines are solid.

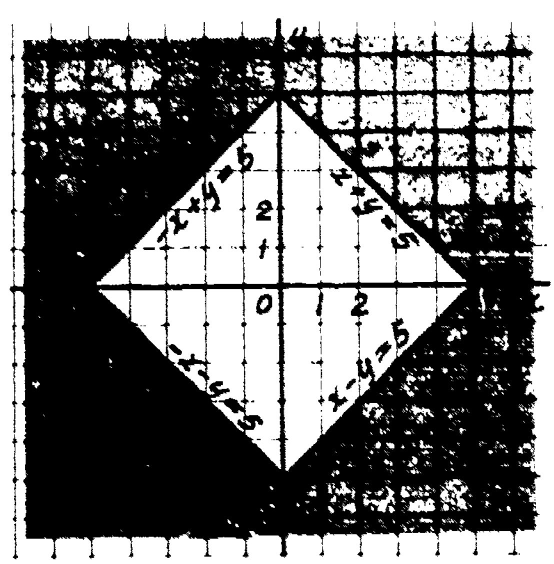


Figure for Problem 2(d)

3.

x	-7	-7	-6	-6	-4	-4	-3	-2	1	3	4	4	7	7
|x|	7	7	6	6	4	4	3	2	1	3	4	4	7	7
|y|	4	4	3	3	1	1	0	-1	-2	0	1	1	4	4
y	4	-4	3	-3	1	-1	0	Impossible		0	1	-1	4	-4

The four open sentences whose graphs form the same figure are:

$x - y = 3$, and $x \geq 3$

$x + y = 3$, and $x \geq 3$

$-x + y = 3$, and $x \leq -3$

$-x - y = 3$, and $x \leq -3$

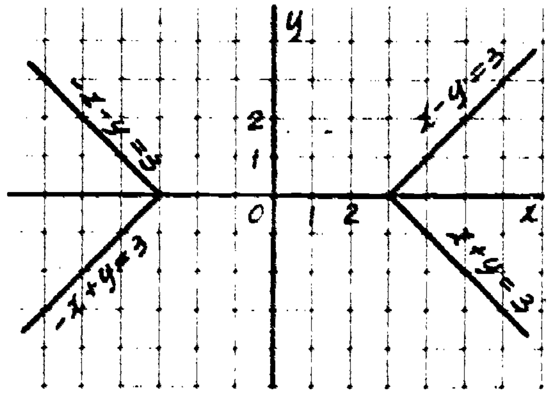


Figure for Problem 3

Answers to Problem Set 3-2c

1. (a)

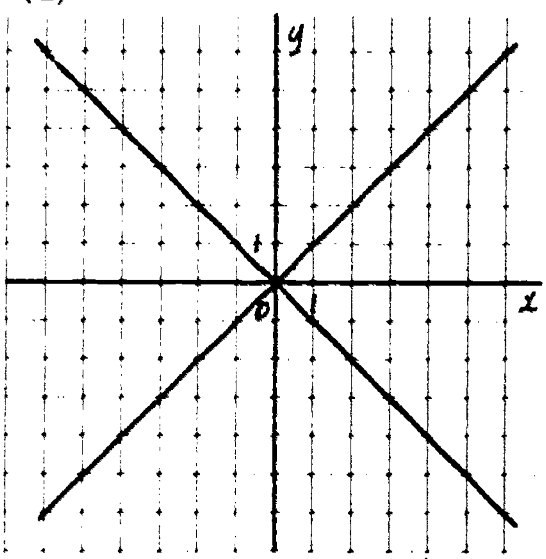


Figure for Problem 1(a)

(b)

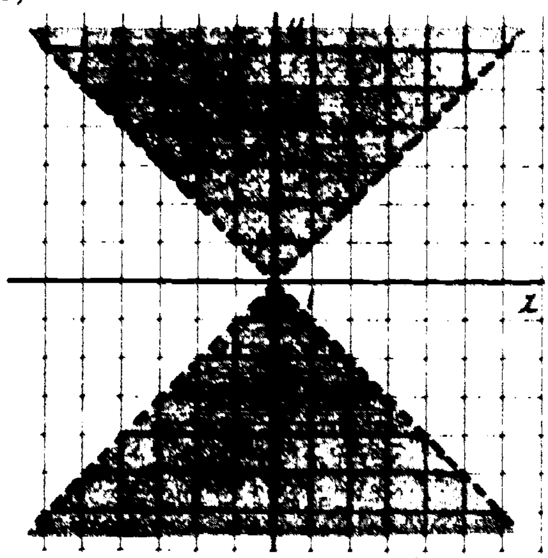


Figure for Problem 1(b)

2. (a)

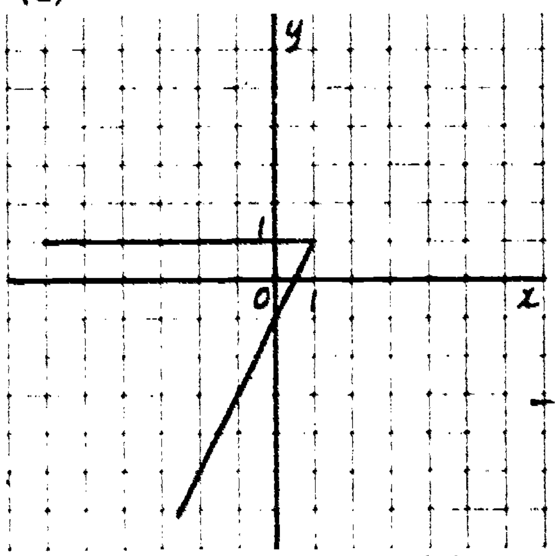


Figure for Problem 2(a)

(c)

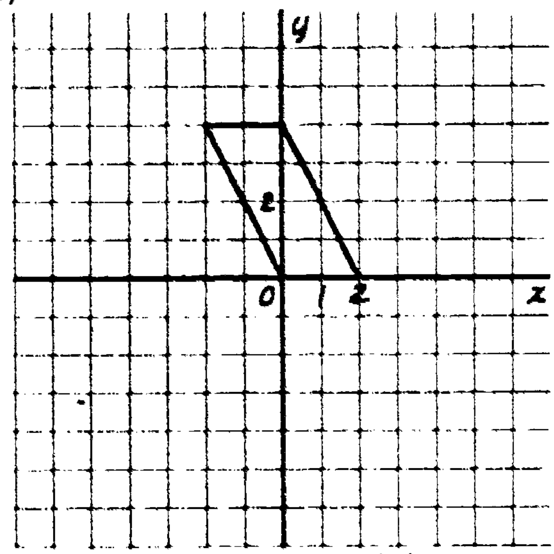


Figure for Problem 2(c)

(b)

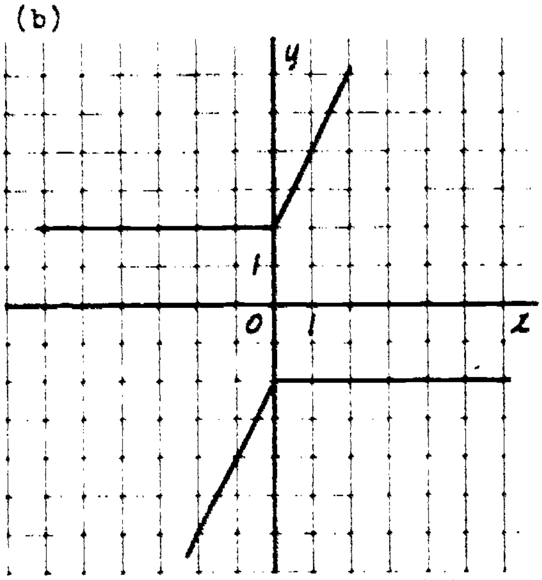


Figure for Problem 2(b)

(d)

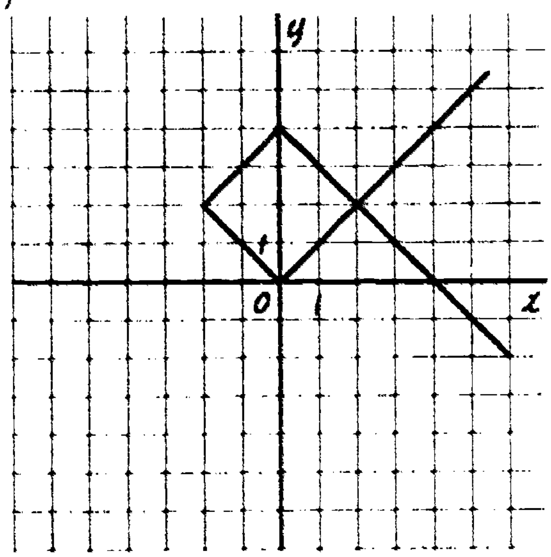


Figure for Problem 2(d)

Chapter 4

ABSOLUTE VALUE AND QUADRATIC EXPRESSIONS

4-1. Absolute Value and Quadratic Equations

The method of solving equations involving absolute value by squaring both members is a convenient alternative to what might be called "the method by definition". It does involve, however, the difficulties which occur when you square both members of an equation.

If $a = b$, then a and b are names for the same number. If that number is squared, a^2 and b^2 are names for the new number and $a^2 = b^2$. It can be proved as follows: If $a = b$ then $a^2 = ab$ and $ab = b^2$; so $a^2 = b^2$ by the transitive property of equality.

This does not work in reverse because there are two square roots of a^2 and of b^2. Thus we could say that $(-3)^2 = (3)^2$, but $-3 \neq 3$.

Consider the following:

Equation	Truth Set
$x = 3$	$\{3\}$
$x^2 = 9$	$\{3,-3\}$

It is apparent that squaring both sides of an equation does not yield an equivalent equation. And yet in solving certain equations involving square roots or absolute values we need to square both sides. We do so then, bearing carefully in mind that we may expect to find a larger truth set in the new equation. We must therefore test the members of this truth set to find which ones really make the original equation true.

Answers to Problem Set 4-1

1. (a) $|2x| = x + 1$

$4x^2 = x^2 + 2x + 1$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$3x + 1 = 0$ or $x = 1$

$x = -\frac{1}{3}$ or $x = 1$

If $x = -\frac{1}{3}$, the left member: $|2(-\frac{1}{3})| = \frac{2}{3}$

the right member: $-\frac{1}{3} + 1 = \frac{2}{3}$

If $x = 1$, the left member: $|2 \cdot 1| = 2$

the right member: $1 + 1 = 2$

The truth set is $\{-\frac{1}{3}, 1\}$.

(b) $x - |x| = 1$

$x - 1 = |x|$

$x^2 - 2x + 1 = x^2$

$-2x + 1 = 0$

$x = \frac{1}{2}$

If $x = \frac{1}{2}$, the left member: $\frac{1}{2} - |\frac{1}{2}| = 0$

the right member: 1

The truth set is ϕ.

(c) $\{1\}$

(d) $\{-1, 7\}$

(e) ϕ ($\frac{1}{3}$ and -1 do not check)

(f) $x^2 - |x| - 2 = 0$

$x^2 - 2 = |x|$

$x^4 - 4x^2 + 4 = x^2$

$x^4 - 5x^2 + 4 = 0$

$x = 1$ or $x = -1$ or $x = 2$ or $x = -2$

1, and -1 do not check. Truth set is $\{2, -2\}$.

2. $|x - 3| = x + 2$

$x^2 - 6x + 9 = x^2 + 4x + 4$

$x = \frac{1}{2}$

The truth set is $\{\frac{1}{2}\}$.

4-2. Solution of Quadratic Inequalities

This material requires considerable understanding of inequalities. The pamphlet "Inequalities" covers this material in detail. The multiplication property of order states:

$$(a < b) \wedge (c > 0) \rightarrow ac < bc$$
$$(a < b) \wedge (c < 0) \rightarrow ac > bc$$

The solution of problems, like $1 \leq |x + 2| \leq 3$, by the method of arguing by cases can become rather tedious if we must use it each time that we want to eliminate an absolute value symbol. It is easier to suffer through the argument once or twice to prove general theorems, which we may then use later without resorting to the two cases of the definition: (i) $0 \leq x$, (ii) $x < 0$, in each problem that we meet. For this reason, we prove the next two theorems.

Suppose $0 < a$. Then

$$|x| \leq a \text{ if and only if } -a \leq x \leq a.$$

Proof. ("Only if") We show first that if $|x| \leq a$, then $-a \leq x \leq a$.

Case (i): If $0 \leq x$, then $|x| = x$ and so if $|x| \leq a$, then $0 \leq x \leq a$; hence $-a \leq x \leq a$, since $-a < 0$.

Case (ii): If $x < 0$, then $|x| = -x > 0$ and so if $|x| \leq a$, then $0 < -x \leq a$, or $-a \leq x < 0$ and since $0 < a$, we can say $-a \leq x < a$.

For the "if" part, we prove

$$\text{If } -a \leq x \leq a, \text{ then } |x| \leq a.$$

Case (i): If $0 \leq x$ and $-a \leq x \leq a$, (i.e., $-a \leq x$ and $x \leq a$), then from $0 \leq x$ and $x \leq a$ it follows that $|x| = x \leq a$ and so $|x| \leq a$.

Case (ii): If $x < 0$ and $-a \leq x \leq a$, then $x = -|x|$, and $-a \leq -|x|$ or $|x| \leq a$.

Suppose $0 < a$. Then $a \leq |x|$ if and only if, either $x \leq -a$ or $a \leq x$.

Proof: ("Only if")

Case (i): If $0 \leq x$ and $a \leq |x|$, then $a \leq x$.

Case (ii): If $x < 0$ and $a \leq |x|$, then $a \leq -x$, or $x \leq -a$

("If")

$0 < a$ and $a \leq x$ give $0 < x$; so $x = |x|$

Therefore, if $a \leq x$, then $a \leq |x|$. Also $0 < a$ (or $-a < 0$) and $x \leq -a$ give $x < 0$; so $x = -|x|$. Therefore, if $x \leq -a$, then $-|x| \leq -a$ or $a \leq |x|$.

Example: Solve $1 \leq |x + 2| \leq 3$

Solution: $1 \leq |x + 2| \leq 3$ if and only if

$$[x + 2 \leq -1 \text{ or } 1 \leq x + 2] \text{ and } [-3 \leq x + 2 \leq 3]$$
$$[x \leq -3 \text{ or } -1 \leq x] \text{ and } [-5 \leq x \leq 1]$$

Combining these cases,

$$[x \leq -3 \text{ and } -5 \leq x \leq 1] \text{ or } [-1 \leq x \text{ and } -5 \leq x \leq 1]$$
$$[-5 \leq x \leq -3] \text{ or } [-1 \leq x \leq 1]$$

The combining of the two cases is actually the use of the Distributive Property of propositional logic: $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$, where $a = (x \leq -3)$, $b = (-1 \leq x)$, and $c = (-5 \leq x \leq 1)$; using $\wedge$ for the conjunction "and" and $\vee$ for the inclusive "or". This point will have to be made by appealing to the student's acceptance of the same meaning for the two cases. Number line diagrams may be helpful to make this point.

We can use the first theorem to give a proof, devoid of case arguments, for the so-called "triangle inequality",

$$|y + z| \leq |y| + |z|$$

Proof: In Problem Set 2-2, No. 6 we proved

$$-|y| \leq y \leq |y|$$
$$-|z| \leq z \leq |z|$$

Hence, adding

$$-(|y| + |z|) \leq y + z \leq |y| + |z|.$$

If we let $x = y + z$, $a = |y| + |z|$, we have by the first theorem

$$-a \leq x \leq a$$

which was proved to be equivalent to

$$|x| \leq a.$$

Or, substituting, $|y + z| \leq |y| + |z|$.

Additional problems, using this method, with answers

1. (a) $|x + 3| < 5$ if and only if $-5 < x + 3 < 5$, i.e., $-8 < x < 2$

   (b) $|3y - 2| \leq 2$ $\qquad 0 \leq y \leq \frac{4}{3}$

   (c) $|4 - m| > 6$ if and only if $4 - m < -6$ or $6 < 4 - m$, i.e. $10 < m$ or $m < -2$

(d) $|2 - p| > 4$ $\qquad$ $p < -2$ or $6 < p$

(e) $\left|\frac{6 - 2x}{3}\right| \geq 2$ $\qquad$ $x \leq 0$ or $6 \leq x$

(f) $-1 \leq |2x - 3| \leq 1$ $\qquad$ $1 \leq x \leq 2$

Answers to Problem Set 4-2

1. $x^2 - 4x + 3 < 0$

   $x^2 - 4x + 4 < -3 + 4$

   $(x - 2)^2 < 1$

   $|x - 2| < 1$

   $x > 1$ and $x < 3$

   The truth set is the set of all numbers such that $1 < x < 3$. It is convenient to use set builder notation here. The answer is then written: $\{x : 1 < x < 3\}$.

2. $\{x : x \leq 1$ or $x \geq 4\}$

3. $x^2 - x < -1$

   $x^2 - x + \frac{1}{4} < -1 + \frac{1}{4}$

   $(x - \frac{1}{2})^2 < -\frac{3}{4}$ $\qquad$ This is impossible. ϕ

4. $x^2 - x > -1$

   $(x - \frac{1}{2})^2 > -\frac{3}{4}$

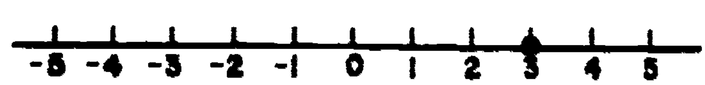
   This is true for all real numbers. The graph is the entire number line.

5. $x^2 - 6x + 9 \leq 0$

   $(x - 3)^2 \leq 0$

   $|x - 3| \leq 0$

   $\{x : x = 3\}$



6. $\{x : -2 < x < \frac{1}{3}\}$

7. $6(-x^2 + 1) > 13x$

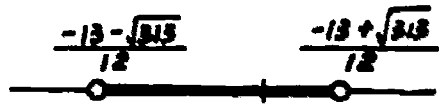
$6x^2 + 13x < 6$

$x^2 + \frac{13}{6}x < 1$

$x^2 + \frac{13}{6}x + \frac{169}{144} < 1 + \frac{169}{144}$

$|x + \frac{13}{12}| < \frac{\sqrt{313}}{12}$

$\{x : \frac{-13 - \sqrt{313}}{12} < x < \frac{-13 + \sqrt{313}}{12}\}$

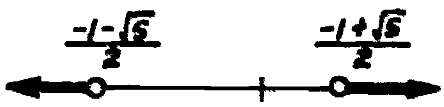


8. $2x - 1 > x - x^2$

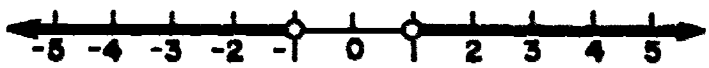
$x^2 + x + \frac{1}{4} > 1 + \frac{1}{4}$

$|x + \frac{1}{2}| > \frac{\sqrt{5}}{2}$

$\{x : x < \frac{-1 - \sqrt{5}}{2}$ or $x > \frac{-1 + \sqrt{5}}{2}\}$



9. $\{x : x < -1$ or $x > 1\}$



10. $\frac{x + 3}{x - 5} > 0$ and $x \neq 5$

Since $(x - 5)^2 > 0$, then $(x - 5)(x + 3) > 0$

$x^2 - 2x - 15 > 0$

$x^2 - 2x + 1 > 16$

$|x - 1| > 4$

$\{x : x < -3$ or $x > 5\}$

11. $|x| > x^2$

$x^2 > x^4$

$x^4 - x^2 + \frac{1}{4} < \frac{1}{4}$

$|x^2 - \frac{1}{2}| < \frac{1}{2}$

$\{x : 0 < x < 1$ or $-1 < x < 0\}$

12. $\{x : x < -1 \text{ or } x > 1\}$

13. $|x| > x + 5$

$x^2 > x^2 + 10x + 25$

$10x < -25$

$x < -\frac{5}{2}$

$\{x : x < -\frac{5}{2}\}$

14. $|2x - 1| > x^2$

If $2x - 1 > 0$, then $2x - 1 > x^2$

$0 > x^2 - 2x + 1$

$0 > (x - 1)^2$

ϕ

If $2x - 1 < 0$, then $-(2x - 1) > x^2$

$0 > x^2 + 2x - 1$

$\{x : -1 - \sqrt{2} < x < -1 + \sqrt{2}\}$

(Note: It is interesting to sketch the graphs of $y = x^2$ and $y = |2x - 1|$ on the same axes and see that this is true.)

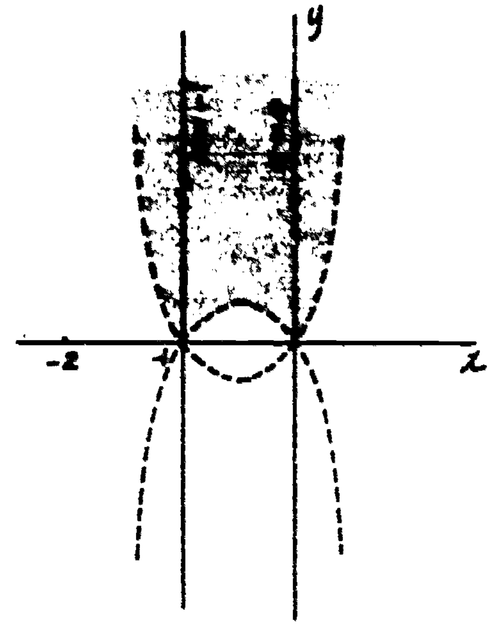
15. Both are equivalent.

16. $|x|(x - 2)(x + 4) < 0$ $\{x : -4 < x < 2, \text{ except } x \neq 0\}$

17. $\frac{|x - 3|}{x - 2} > 0$. Since $|x - 3| > 0$, $x - 2 > 0$, but $x - 3$ cannot be equal to 0. $\{x : x > 2, \text{ except } x \neq 3\}$

Answers to Problem Set 4-3

1. $|x^2 + x| < y$



Graph $y = |x^2 + x|$

Other than those values where $x^2 + x = 0$, consider $x^2 + x > 0$ or $x^2 + x < 0$.

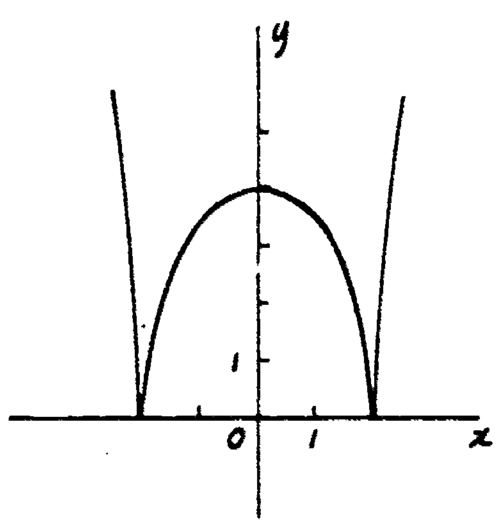
(1) If $x^2 + x > 0$, then $|x^2 + x| = x^2 + x$. So, the graph of $y = x^2 + x$ is the parabola. Hence, the graph of $y > x^2 + x$ is above this parabola and in the domain $x^2 + x > 0$, i.e., $x > 0$ or $x < -1$. For, $x(x + 1) > 0$, then $[x > 0$ and $x + 1 > 0]$ or $[x < 0$ and $x + 1 < 0]$. $\{x : x < -1$ or $x > 0\}$.

(2) If $x^2 + x < 0$, then $x(x + 1) < 0$, or $[x < 0$ and $x + 1 > 0]$ or $[x > 0$ and $x + 1 < 0]$. $\{x : -1 < x < 0$ or $\phi\}$

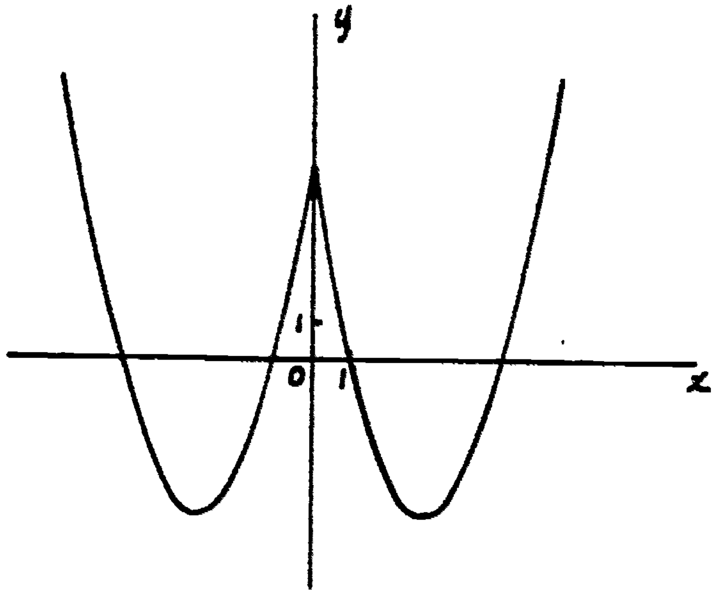
The union gives the interval, $-1 < x < 0$. So, the graph of the solution set of $y > -(x^2 + x)$ in the domain $x^2 + x < 0$ is the region above the parabola and between $x = -1$ and $x = 0$.

$\therefore$ The graph of the solution set of $|x^2 + x| < y$ is the union of these two regions and $x = -1$, $x = 0$.

2.



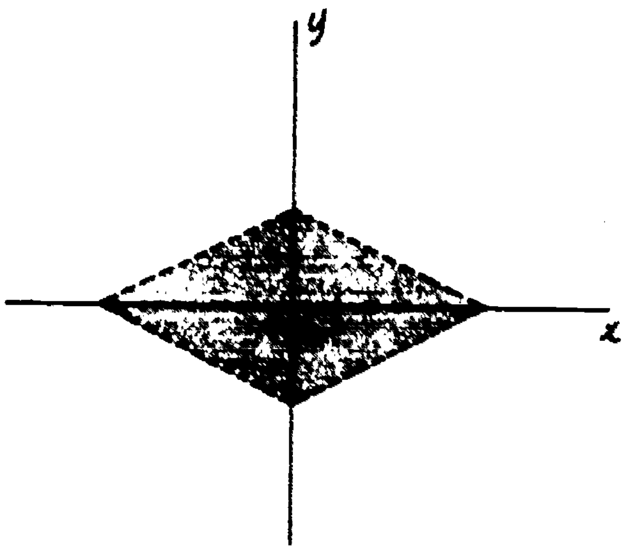
3.



We recall here that $y = x^2 - 6|x| + 5$ implies:

$$y = \begin{cases} x^2 - 6x + 5, & x \geq 0 \\ x^2 + 6x + 5, & x < 0 \end{cases}$$

4.



Chapter 5

ABSOLUTE VALUE, COMPLEX NUMBERS AND VECTORS

5-1. Absolute Value of a Complex Number

If the student has had no work in complex numbers he should not attempt to do this section. In many Algebra II texts, however, there is an introduction to the topic of complex numbers. If the student has had the application of the four fundamental operations to the complex numbers, he might well continue with this section. Another pamphlet of this series entitled "Complex Numbers" is devoted entirely to this topic and includes most of the material in this section.

The representation of complex numbers by points in the plane had a great effect historically on the acceptance of the complex number system by mathematicians. This geometric representation overcame the feeling that the complex number system was not concrete. It was found that the complex number system could be employed in the solution of geometric problems.

The notion of absolute value is a purely algebraic one, even though its definition is geometrically motivated. All of the properties of absolute value can be established algebraically.

The real, nonnegative number $\sqrt{a^2 + b^2}$ is called the absolute value (or modulus) of the complex number, and can be written $|a + bi|$. The angle θ associated with the number $a + bi$ is called the argument (or amplitude) of $a + bi$.

Answers to Problem Set 5-1

1. (a) 5
   (b) 2
   (c) 0
   (d) $\sqrt{2}$
   (e) $\sqrt{\pi^2 + 2}$

2. Let $z = x + yi$, then $\frac{z}{|z|} = \frac{x + yi}{\sqrt{x^2 + y^2}}$, and

$$\left|\frac{z}{|z|}\right| = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2}} = 1$$

3. (a) The single point $(1,0)$

(b) Let $z = x + yi$, x and y real

Then $x + yi = \sqrt{x^2 + y^2}$. Since $\sqrt{x^2 + y^2}$ is real, $y = 0$. Hence, $x = \sqrt{x^2}$. By definition, the truth set is the set of points comprising the non-negative x-axis.

(c) Since z cannot be zero, the given equation may be transformed into the equation $|z| = 1$, and this is the equation of the unit circle.

4. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

Then
$$\begin{aligned}
|z_1 z_2| &= |(x_1 + y_1 i)(x_2 + y_2 i)| \\
&= |(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i| . \\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1x_2y_1y_2 + x_2^2y_1^2} \\
&= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)} \\
&= \sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2} \\
&= |z_1||z_2|
\end{aligned}$$

5. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

Then
$$\frac{z_1}{z_2} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2} = \frac{(x_1x_2 + y_1y_2) + (x_2y_1 - x_1y_2)i}{x_2^2 + y_2^2}$$

and
$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \frac{\sqrt{x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_2^2y_1^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2}}{x_2^2 + y_2^2} \\
&= \frac{\sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}}{x_2^2 + y_2^2} \\
&= \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{|z_1|}{|z_2|}
\end{aligned}$$

6. Using the fact that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side, we have

$$|z_1 - z_2| + |z_2| \geq |z_1| \quad \text{and} \quad |z_1 - z_2| + |z_1| \geq |z_2|$$

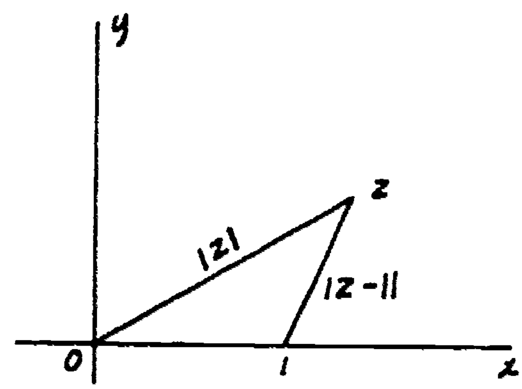
or

$$|z_1 - z_2| \geq |z_1| - |z_2| \quad \text{and} \quad |z_1 - z_2| \geq |z_2| - |z_1|$$

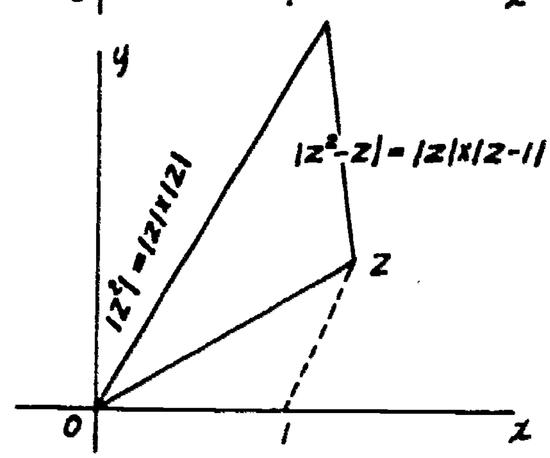
From this we conclude

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

7. The triangle with vertices $0, 1, z$ is shown in the figure at the right. The lengths of the sides of the triangle are $1, |z|, |z - 1|$.



If we multiply each of these lengths by $|z|$, we obtain $|z| \cdot 1$, $|z| \cdot |z|$, $|z| \cdot |z - 1| = |z^2 - z|$. These are the lengths of the sides of a triangle whose vertices are $0, z, z^2$ as the second figure clearly shows.



The two triangles are similar because corresponding sides are proportional. To obtain a geometric construction for z^2, one must choose a unit of length on the x-axis, draw a triangle with vertices $0, 1, z$, and then construct a second triangle similar to the first one by making each side of the second triangle $|z|$ times as long as the sides of the first. The vertex of the second triangle which corresponds to z of the first triangle is z^2.

5-2. Complex Conjugates

This section is concerned with those aspects of the complex conjugate which involve absolute value. A few preliminary remarks are made about the conjugate of a complex number. It is assumed that the student will have had some other experience with conjugates.

The introduction of the notion of complex conjugates has several important consequences. It makes possible the simplification of computations involving absolute values and multiplicative inverses; the algebraic representation of the geometric operation of reflection in a line; the algebraic formulation and manipulation of statements involving the real and imaginary parts of the complex number; the algebraic representation of geometric relations in terms of complex numbers.

Answers to Problem Set 5-2

1. $-(2 - 3i) = -2 + 3i$

   $(\overline{2 - 3i}) = 2 + 3i$

   $|2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$

   $|\overline{2 - 3i}| = |2 + 3i| = \sqrt{13}$

   $\dfrac{1}{2 - 3i} = \dfrac{\overline{2 - 3i}}{|2 - 3i|^2} = \dfrac{2 + 3i}{13} = \dfrac{2}{13} + \dfrac{3}{13}i$

   $|2 - 3i|^2 = (\sqrt{13})^2 = 13$

   $|(2 - 3i)^2| = |2 - 3i|^2 = 13$

   $\dfrac{4 + 5i}{2 - 3i} = (4 + 5i)\dfrac{1}{2 - 3i} = (4 + 5i)(\dfrac{2}{13} + \dfrac{3}{13}i) = -\dfrac{7}{13} + \dfrac{22}{13}i$

2. (a) $\frac{3}{2} + (-\frac{1}{2})i$ (e) $-\frac{3}{25} + \frac{46}{25}i$

   (b) $\frac{1}{10} + \frac{3}{10}i$ (f) $\frac{1}{2} + (-\frac{3}{4})i$

   (c) $-\frac{1}{13} + (-\frac{5}{13})i$ (g) $-\frac{9}{25} + (-\frac{38}{25})i$

   (d) $\frac{7}{29} + \frac{26}{29}i$ (h) $-3 + (-\frac{3}{2})i$

3. (a) circle of radius 3 with center at $(2,0)$

   (b) set of points exterior to circle of radius 3 with center at $(-2,0)$.

   (c) set of points interior to circle of radius 4 with center at $(0,2)$.

   (d) set of points interior to, or on, circle of radius 5 with center at z_0.

4. $|x + yi - (2 + 3i)| = 5$

$|(x - 2) + (y - 3)i| = 5$

$\sqrt{(x - 2)^2 + (y - 3)^2} = 5$

$(x - 2)^2 + (y - 3)^2 = 25$

$x^2 + y^2 - 4x - 6y - 12 = 0$

The set of points satisfying the given equation is the circle of radius 5 with center at $(2,3)$.

5. Since $y^2 \geq 0$, $x^2 + y^2 \geq x^2$. Hence, $\sqrt{x^2 + y^2} \geq \sqrt{x^2}$. This gives $|x + iy| \geq |x| \geq x$, therefore, $|z| \geq x$.

6. $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 - \bar{z}_1 z_2 - z_1\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$$

$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + \bar{z}_1 z_2 + z_1\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$$

Thus, $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

7. It is sufficient to show that $\left|\frac{z_1}{z_2}\right|^2 = \frac{|z_1|^2}{|z_2|^2}$.

But $\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\left(\overline{\frac{z_1}{z_2}}\right) = \left(\frac{z_1}{z_2}\right)\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{z_1\bar{z}_1}{z_2\bar{z}_2} = \frac{|z_1|^2}{|z_2|^2}$

8. (a) The distance from the origin of z_1 is less than that of z_2.

(b) z is on the circle of radius 5 with center at the origin.

9. If $z = x + yi$ the stated conditions become $x = y$, $\sqrt{x^2 + y^2} = 1$ the solutions of this pair of equations are $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ and $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}$. The solutions of the problem are, therefore,

$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

10. The point z belongs to the set if and only if $|z - \bar{z}_0| < |z - z_0|$, that is, if and only if the distance from z to $\bar{z}_0$ is less than the distance from z to z_0. This will be true if and only if the point z lies on the same side as $\bar{z}_0$ of the perpendicular bisector of the line segment joining z_0 and $\bar{z}_0$. This perpendicular bisector is the x-axis. Thus the set is the set of all points z which lie on the same side of the x-axis as $\bar{z}_0$. This can also be established by calculation.

11. The proposition is true provided x and y are real. In this event we have

$$|x| + |y| \leq \sqrt{2}\,|z| \qquad \text{if and only if}$$
$$(|x| + |y|)^2 \leq 2|z|^2$$

Now $|z|^2 = x^2 + y^2$ and we have $|x|^2 + 2|x||y| + |y|^2 \leq 2|x|^2 + 2|y|^2$ this reduces to $0 \leq |x|^2 - 2|x||y| + |y|^2$ or $0 \leq (|x| - |y|)^2$ which is true because the square of any real number is non-negative. Q.E.D. The proposition is not true for all complex values of x and y as the following counter example will show.

Let $x = 8 + 2i$ and $y = -1 + 4i$, then $|x| = \sqrt{68} = 2\sqrt{17}$ and $|y| = \sqrt{17}$

$$|x| + |y| = 3\sqrt{17}$$

$z = x + yi = (8 + 2i) + i(-1 + 4i) = 4 + i$. So $|z| = \sqrt{17}$. It is false that $\sqrt{2}\,\sqrt{17} \geq 3\sqrt{17}$, hence in this case $|x| + |y|$ is not equal to or less than $\sqrt{2}\,|z|$.